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PARAPOTENTIAL ELECTRON FLOW AND THE VACUUM PINCH

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I. INTRODUCTION

For purposes of discussion we define orthopotential electron flow as a flow which crosses electrostatic equipotentials normally. This is the flow which dominates in customary diode regions at low currents and energies. We call the corresponding flow along equipotentials parapotential, and it is implied that this is the kind of flow which is dominant in high-current, relativistic situations. From the evidence presently at hand it appears that this is a justifiable interpretation, although any real flow shares some of both properties. In this general context parapotential flow may take place either normally to the magnetic field or at some other angle; in beams such oblique angles are to be looked for in the presence of a longitudinal magnetic field. The resulting shear motion under these circumstances considerably complicates the analysis without apparently adding anything essential, and in what follows we shall take parapotential flow to be always perpendicular to both $\underline{E}$ and $\underline{H}$, which themselves will always be considered orthogonal. No equations of motion are written for this flow, the electron being always assumed under no force. Centrifugal forces due to abrupt trajectory changes are thus not allowed for.

II. DERIVATION OF THE PARAPOTENTIAL FLOW EQUATION IN AXIAL SYMMETRY

We derive an equation to describe equilibrium parapotential flow in axial symmetry, which is the most important case under present consideration. We define for reference purposes a constant-z plane on which there is a normally incident, radius-dependent current density J (see figure 1); the radius on which this current is injected is ρ. At some other point this same current is at another radius r, and therefore the current density at this point is $J(\rho/r)(d\rho/\delta)$ from the figure. We also have $\delta = dr \cos\phi$; since there is an implied functional dependence $\rho = \rho(r,z)$, we have

$$-\tan\phi = dr/dz = -(\partial\rho/\partial z)/(\partial\rho/\partial r).$$

Therefore

$$d\rho/\delta = (\rho_r^2 + \rho_z^2)^{\frac{1}{2}},$$

where the subscripted quantities are the appropriate partial derivatives, and so the scalar current density is given by

$$en\beta = J(\rho/r)(\rho_r^2 + \rho_z^2)^{\frac{1}{2}}, \qquad (1)$$

A fundamental requirement for parapotential flow is $E = \beta H$, which gives

$$\beta = (r/2I)(V_r^2 + V_z^2)^{\frac{1}{2}}, \qquad (2)$$

where V is the potential and I is the current contained within that potential; I is given by

$$I = \int_0^{\rho} 2\pi\rho' \, J(\rho') \, d\rho'. \qquad (3)$$

From (1), (2), (3), and Poisson's equation

$$\nabla^2 V = 4\pi e n, \qquad (4)$$

we get

$$\nabla^2 V = 4\,(dI/dr)\,r^{-2}(\rho_r{}^2 + \rho_z{}^2)^{\frac{1}{2}}(V_r{}^2 + V_z{}^2)^{-\frac{1}{2}};$$

if use is now made of the fact that ρ is a function of V only (a consequence of parapotentiality), this can be written

$$\nabla^2 V = 4r^{-2} I(dI/d\rho)(d\rho/dV)$$

or simply

$$\nabla^2 V = 2r^{-2} dI^2/dV, \qquad (5)$$

which is the fundamental equation of the flow. The requirement $E = \beta H$ must be imposed after the first integration of (5), which introduces an arbitrary function; this requirement also helps to define the unspecified function of V, dI^2/dV.

III. SOLUTION OF THE PARAPOTENTIAL EQUATION WITH $\partial/\partial z \ll \partial/\partial r$ or $\partial/\partial r \ll \partial/\partial z$

It will not be attempted here to find a general solution of (5). Instead we shall be content with solutions under the asymptotic conditions in which either the radial or axial terms of $\nabla^2 V$ can be neglected. We consider first the case where $\partial/\partial z \ll \partial/\partial r$, having first written (5) in the more convenient form

$$\nabla^2 \gamma = 2r^{-2}\, d\mu^2/d\gamma, \qquad (5a)$$

γ being the usual total electron energy in mc^2 units and μ being the current in units of 17 kA (by analogy with ν, which is μ/β). Then we approximate (5a) by

$$r^{-1}\partial/\partial r(r\partial\gamma/\partial r) = 2r^{-2}\, d\mu^2/d\gamma. \tag{5b}$$

On integrating once we have

$$(\partial\gamma/\partial r)^2 = 4r^{-2}(\mu^2 - \mu_o^2), \tag{6}$$

where μ_o is in general an arbitrary function of z. In the present units the condition E = βH becomes

$$\partial\gamma/\partial r = 2\beta\mu/r,$$

so that $\beta^2 = 1-\gamma^{-2} = 1-\mu_o^2/\mu^2$, or

$$\gamma = \mu/\mu_o. \tag{7}$$

and $d\mu^2/\partial\gamma = 2\mu_o^2\gamma$; application of E=$\beta$H after the first integration has determined the RHS of (5b) and has shown this equation to be linear. From (6) and (7) we have after integrating

$$\gamma = \cosh\left[2\mu_o \log (r/r_o)\right], \tag{8}$$

which is the complete solution, involving the two arbitrary functions of , μ_o and r_o. On $r=r_o$ the electron energy and the electric field are zero. Accordingly the interpretation of μ_o is that it is a current which must flow _within_ the cathode surface to produce the required conditions of the flow. In a solution by Rostoker[1] of the transport problem in a medium there is plasma present which can neutralise the beam. The vacuum flow is likewise possible with an internal metallic conductor, and this is the case with which we shall be principally concerned. (An H_z field can presumably also be employed in this way, but this situation will not be considered). The only interesting form of solution is that for which $\mu_o(z)$ is constant. This provides a γ which depends only on r/r_o, so that the whole flow can be tapered like a coaxial transmission line at constant impedance.

If μ_o is larger than that value which just satisfies (8) with $r=r_a$ (the anode radius) and $\gamma=\gamma_a$ (the applied anode potential), the beam assumes a maximum radius r_b and energy γ_b satisfying

$$\gamma_b = \cosh (2\mu_o \log r_b/r_o), \tag{9a}$$

$$(\gamma_a - \gamma_b)/(r_b \log r_a/r_b) = (2\mu_o/r_b) \sinh (2\mu_o \log r_b/r_o), \tag{9b}$$

these being required to satisfy the potential and field continuity conditions at the beam surface. If μ_o is less than this critical value, the flow is impossible. If the radii are large compared to the conductor separation, we can write approximately

$$\gamma_a \sim 4\mu_o{}^2\sigma(\tau-\sigma) + 1 + 2\mu_o{}^2\sigma^2, \ \sigma \equiv r_b/r_o - 1, \ \tau \equiv r_a/r_o - 1,$$

whence

$$\mu_o{}^2 = (\gamma_a - 1)/2\sigma(2\tau - \sigma). \tag{10}$$

The critical current obtains for $\sigma=\tau$; thus

$$\mu_c{}^2 = (\gamma_a - 1)/2\tau^2. \tag{11}$$

There is a minimum of (10) at $r=r_1$; this corresponds to $r_b = r_a$, or $\gamma_b = \gamma_a$. The maximum energy is driven downward together with r_b as μ_o is increased beyond μ_c. It is clear from (7) that at extreme relativistic energies the contribution of μ_o to the total current is small. In an actual flow with a central conductor the required level of μ_o is likely to be supplied by orthopotential flow from the end of the conductor. Presumably the first part of this conductor serves as an emitter, injecting the necessary current by space-charge-limited field emission along initially orthopotential trajectories (figure 2).

The volt-ampere characteristic with $\mu = \mu_c$ is

$$I/V_a = \mu/(\gamma_a - 1) = 2(1 - 1/\gamma_a)^{-1/2} \text{ arc cosh } \gamma_a/\log(r_a/r_c). \tag{12}$$

In the nonrelativistic limit this is

$$I/V_a \to 2\sqrt{2}\,(\gamma_a - 1)^{1/2}/\log(r_a/r_o); \tag{13}$$

in the extreme relativistic limit it is

$$I/V_a \to 2 \log 2\gamma_a/\log(r_a/r_o). \tag{14}$$

These are in Gaussian units, where I is the total current and V_a is the applied potential; the conductance in mhos is 1/30 of these values. The internal current is included in this calculation since it is generally supplied by the same source as the sheath current and results in electrons at the same energy. In terms of the current actually delivered at the end of the flow it is not particularly important (except as the angular distribution may be affected) whether it originates far back on the cathode or at the end. In any case the magnetic insulation provided by the central current should prevent any electron transport across the inter-electrode space in the cylindrical region. As has been pointed out, the high conductance in the nonrelativistic case is principally due to the core current μ_o; in the extreme relativistic case the paraequipotential sheath carries most of the current. The minimum conductance (at $\mu_o = \mu_c$) occurs for $\gamma_a \approx 2.5$, and the corresponding conductance is

$$5.22/\log(r_a/r_c) \text{ gaussian units} = 0.174/\log(r_a/r_c) \text{ mhos.} \tag{15}$$

This minimum is very broad, and little error is incurred over the range 250 kev $< V_a <$ 2 Mev if this figure is used.

Conditions at the end of the central conductor cannot be adequately described by an equation which neglects the z-derivatives in (5a), and we therefore consider the other extreme case, for which $\partial/\partial r \ll \partial/\partial z$ and which gives the equation

$$\partial^2\gamma/\partial z^2 = 2r^{-2}\, d\mu^2/d\gamma. \tag{5c}$$

The solution proceeds just as in the previous case. Thus as the first integral we get

$$(\partial\gamma/\partial z)^2 = 4(\mu^2 - \mu_o^2)$$

with the condition E=βH dictating as before $\gamma=\mu/\mu_o$, where now μ_o is a function of z. The complete solution is

$$\gamma = \cosh[2\mu_o(B - z/r)], \tag{16}$$

R being the other arbitrary function of r. The quantities μ_o and R depend on the cathode and beam surface shapes; thus if the cathode is at $z_o(r)$ and the edge of the beam at $z_b(r)$, then $\dot{z}_o = Rr$ and

$$z_b - z_o = (r/\mu_o) \text{ arc cosh } \gamma_b, \tag{17}$$

In the case of a metallic cathode μ_o is constant, so that a constant γ_b implies $z_b - z_o \propto r$. This is an obvious limiting case of the coaxial flow previously discussed, in which the separation of the equipotentials is simply proportional to r (figure 4). Accordingly for metallic cathodes the flow presents no new aspects. The ratio d_o/r_o (figure 3) continues to play the role of an impedance, which can thus be taken as the order of

$$5.74 \; d_o/r_o \text{ ohms} \tag{18}$$

from (15) calculated at the minimum conductance. This concept furnishes a convenient way of matching the diode to a given source impedance. The arbitrary taper is likewise useful in concentrating the flow and confining the angular spread of the electrons.

IV. THE PARALLEL-PLANE VACUUM-PINCH DIODE

The case of the parallel-plane vacuum-pinch diode cannot be treated in terms of these simple models since there is no conical metallic cathode, and the coincidence of flow lines and equipotentials can therefore not be satisfied throughout the flow. It is presently conjectured, however, that the flow in this tube is divided into an orthopotential core and a parapotential sheath, the configuration of the latter being such as to satisfy approximately the conditions derived for the case of a conical metallic cathode down to some suitable potential, say V_1. Current is then injected into the parapotential portion from an outside ring, as usual (figure 4). The potential V_1 is taken as considerably below V_a, so that it gives rise to a roughly conical core. This core provides the required $\mu_o(r)$ to satisfy the $E=\beta H$ condition in the sheath. The regulatory mechanism for this consists in the self-adjustment of the potential V_1 so as to provide suitable Child's-law conditions at the cathode, and we should therefore expect that the overall conductance is given in order of magnitude by the naive expression which equates the interelectrode separation with the maximum penetration of an electron in the approximately constant electric field V_a/d and magnetic field $2I/r_o$; the resulting conductance in mhos can easily be shown to be

$$0.017\ (1 + 1/V_{MV})^{\frac{1}{2}}(r_o/d), \qquad (19)$$

where V_{MV} is V_a expressed in megavolts. Because of the orthopotential character of the core flow it is expected that core electrons will cross the sheath trajectories, complicating the description of the flow. These transluminar core electrons should provide a halo around the central spot which should become less important as the energy increases and less core current is required to satisfy parapotential conditions in the sheath. It seems hopeless, however, to try to treat this flow accurately except with a computer.

REFERENCE

1. Norman Rostoker, private communication.

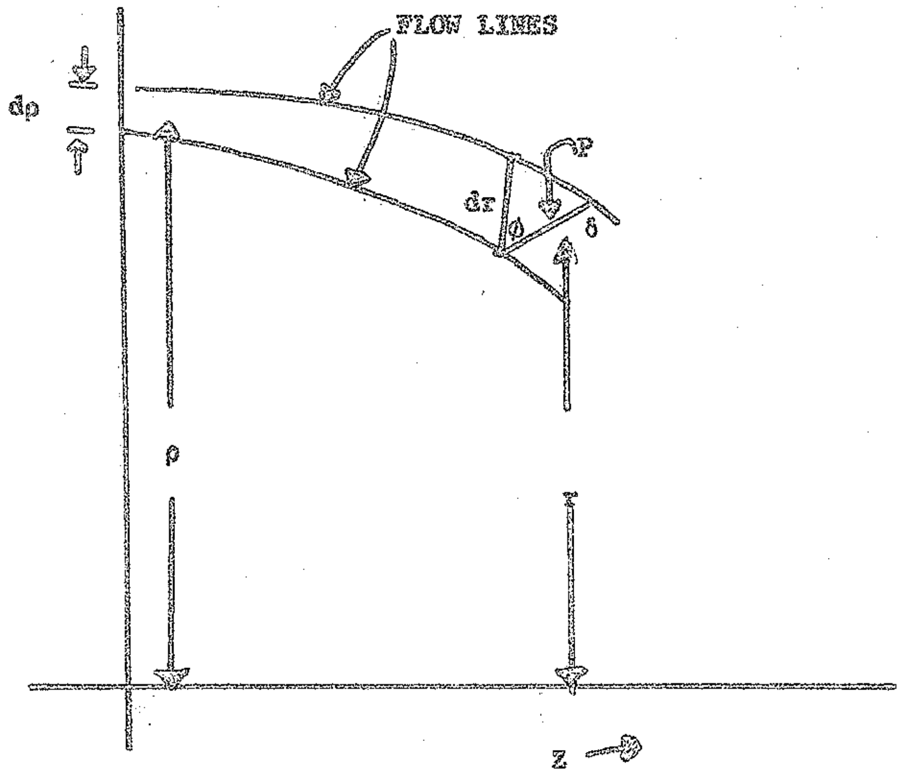

Figure 1: Axially symmetric parapotential flow.

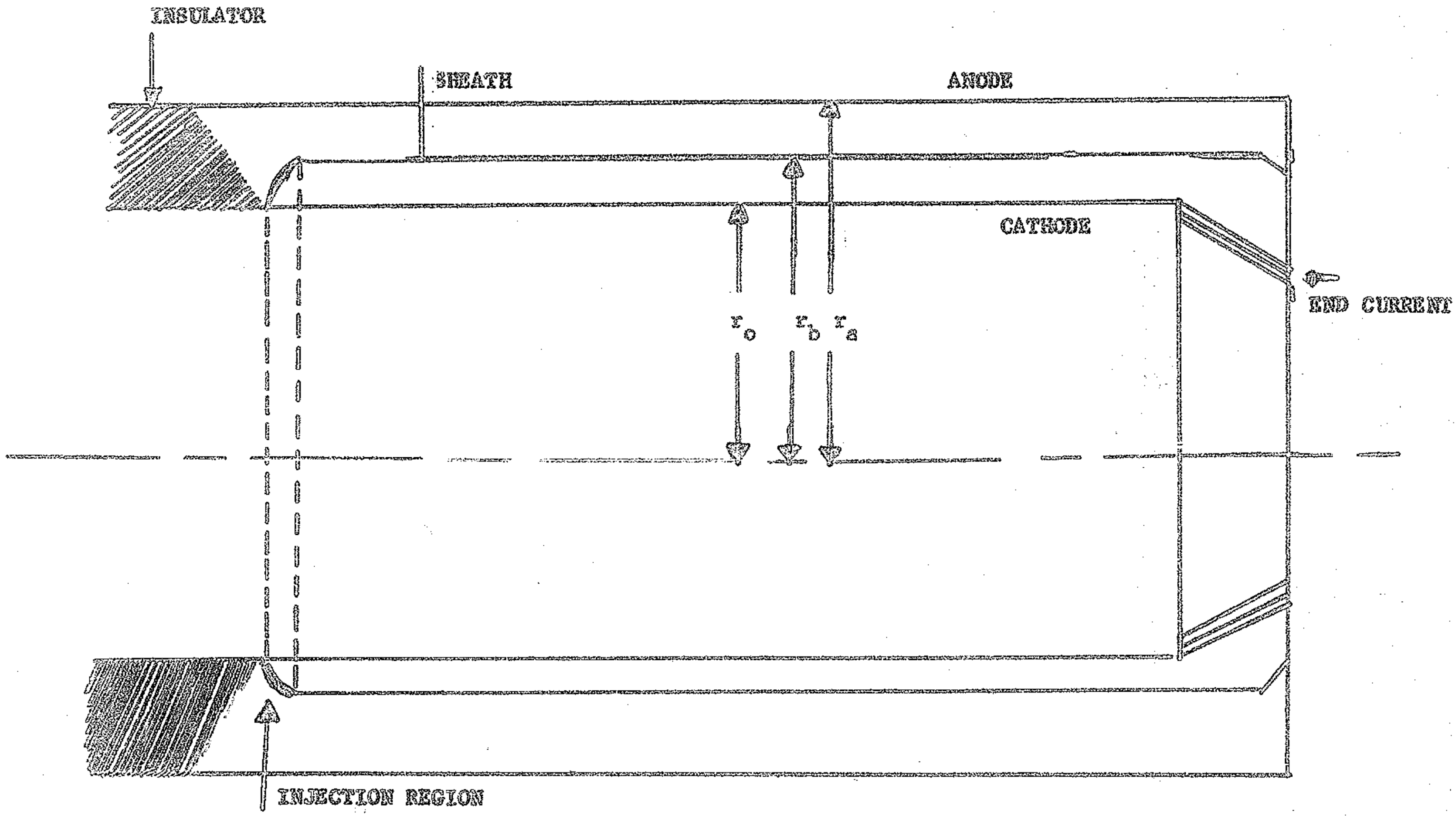

Figure 2: Schematic of coaxial parapotential flow

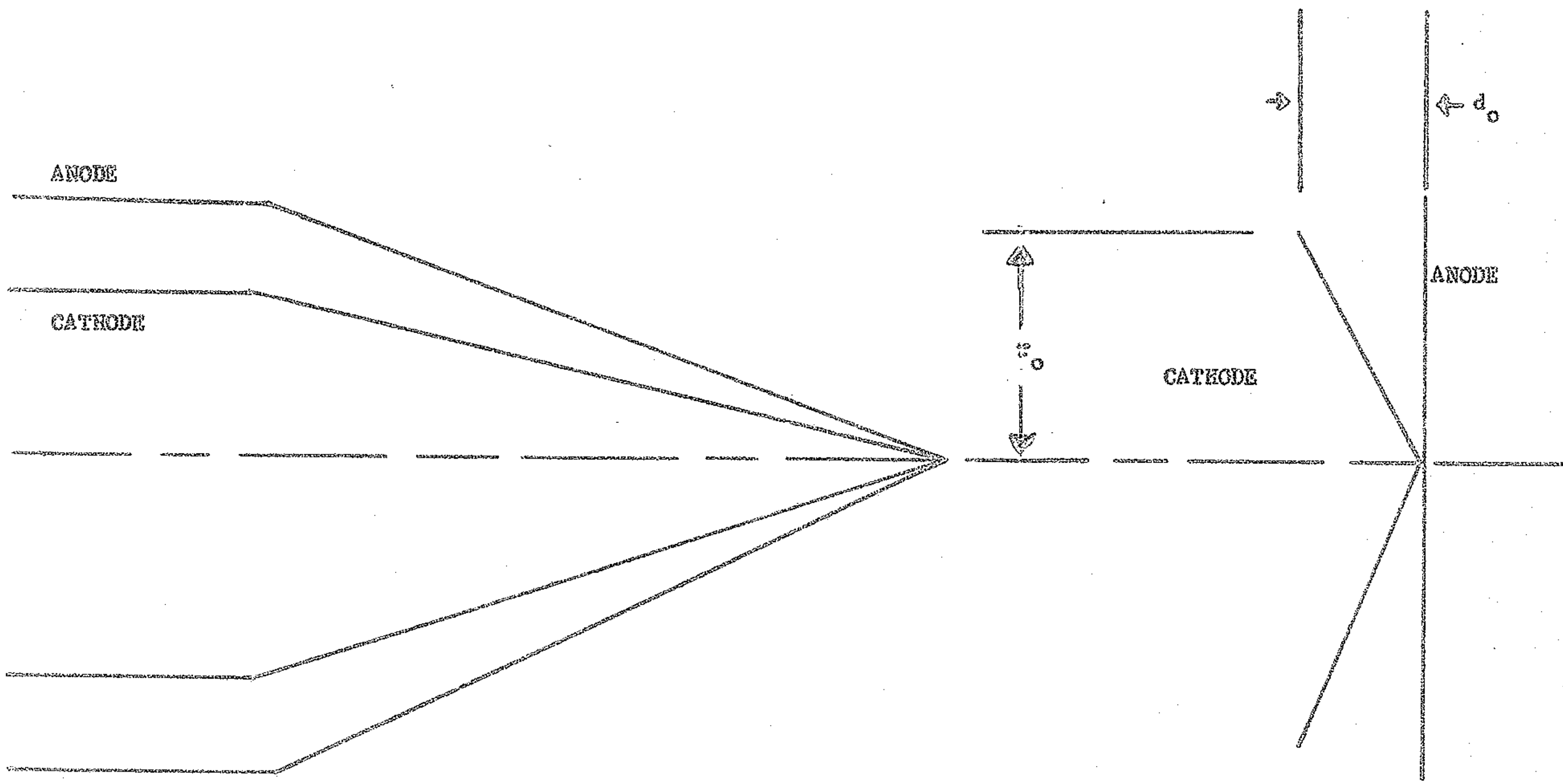

Figure 3: Gently and sharply tapered configurations of roughly equal impedance.

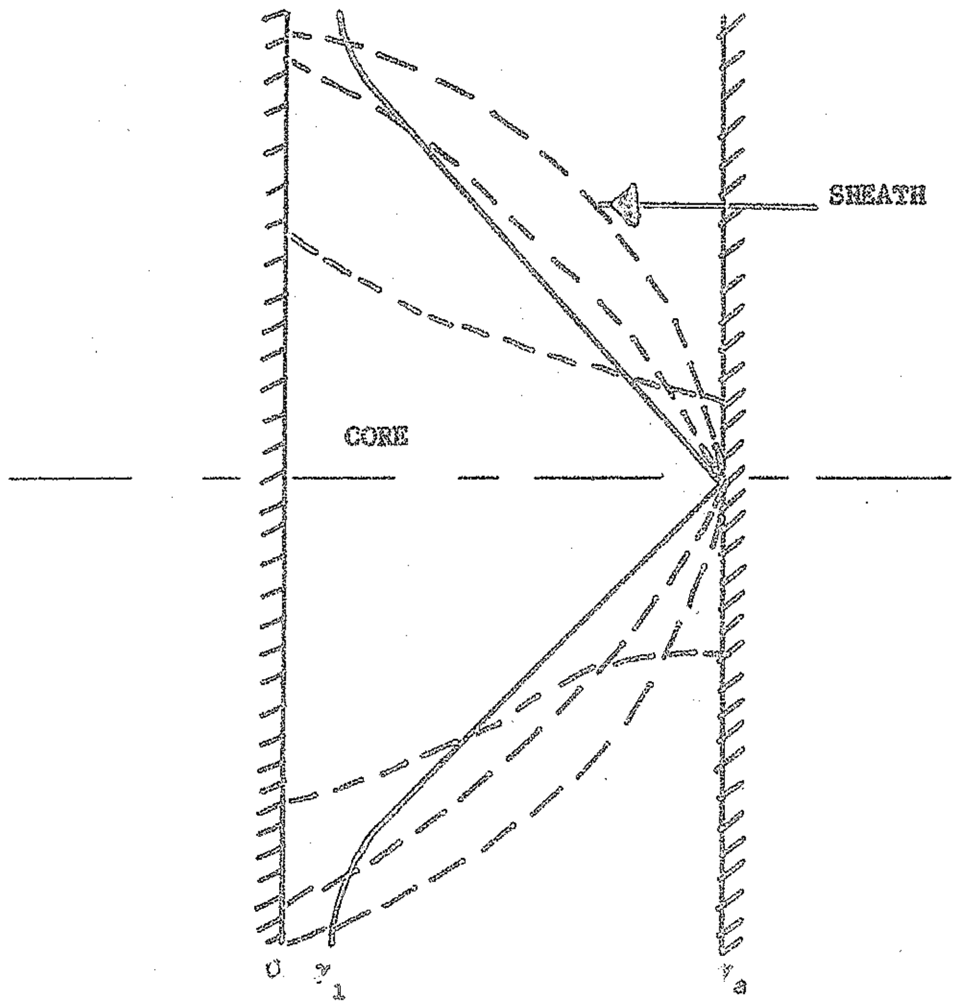

Figure 4: Suggested flow pattern in parallel-plane, vacuum-pinch diode. Solid lines are equipotentials; dashed lines are trajectories.